The Distance Spectra of Cayley Graphs of Coxeter Groups

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Outline

Root Systems

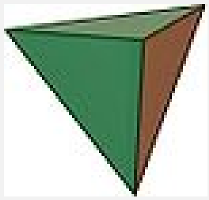
- V an n dimensional inner product space over $\mathbb{R}$
- Given $\alpha \in V$, reflection $t_\alpha : V \to V$ fixes hyperplane $H_\alpha := \{v \in V : (\alpha, v) = 0\}$ (pointwise) and sends α to $-\alpha$.
- Let $\Phi \subset V$ satisfy
  - $t_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$
  - $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\}$ for all $\alpha \in \Phi$
- Φ is called a root system.

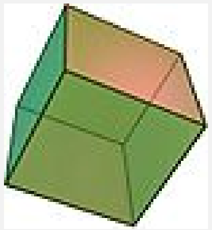
Finite Reflection Group

- Let Φ be an irreducible finite root system
- Let W be the group generated by reflections t_α, $\alpha \in \Phi$
- W called a finite reflection group or finite Coxeter group
- Four infinite families: A_n, B_n, D_n, $\mathsf{I}_2(m)$, and six exceptional types: E_6, E_7, E_8, F_4, H_3, H_4

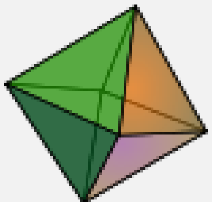
Reflection Groups Arising in Nature

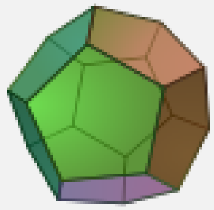
$\mathfrak{S}_4 = W(\mathsf{A}_3)$



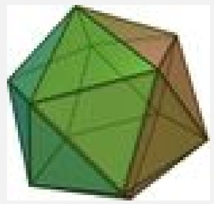


$W(\mathsf{B}_3)$





$W(\mathsf{H}_3)$

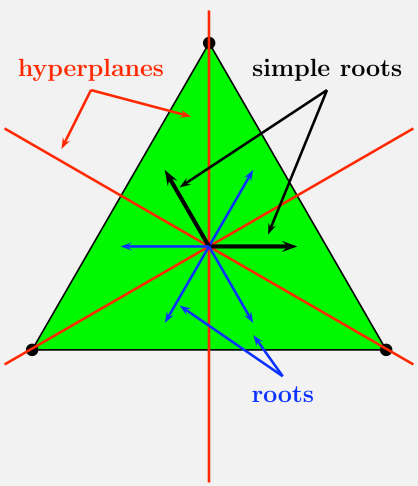


Source: http://en.wikipedia.org/wiki/Regular_polytope

Simple Systems

- $\Delta \subset \Phi$ is a set of simple roots if:
  - Δ is a basis for V
  - Every root $\alpha \in \Phi$ can be written $\alpha = \sum_{\alpha_i \in \Delta} c_i \alpha_i$ where all $c_i \geq 0$ or all $c_i \leq 0$
- The simple reflections are $S := \{s_\alpha : \alpha \in \Delta\}$

Ex: $\mathfrak{S}_3 = W(\mathrm{A}_2)$–Symmetries of a triangle



Length and Absolute Length

- Let $T := \{$set of all reflections$\}$
- Let $S := \{$set of simple reflections$\}$
- Every element $w \in W$ is a word in the reflections in T or S
- The minimum length $\ell_T(w)$ of $w \in W$ written as a word in T is called the absolute length of w.
- The minimum length $\ell_S(w)$ of $w \in W$ written as a word in S is called the length of w.
- Note: $\ell_T(w)$ is constant on conjugacy classes, while $\ell_S(w)$ is not

Coxeter Transformations

- The Coxeter element of W is $c = \prod_{s \in S} s$ (unique up to conjugacy)
- The order of c is h, the Coxeter number
- The eigenvalues of c (in the reflection representation) are of the form ζ^{m_i} for some primitive h^{th} root of unity ζ.
- The numbers $m_1, m_2, \dots m_n$ are the exponents of W

The Exponents of the Reflection Groups

Type	$m_1, m_2, \dots, m_n$
A_n	$1, 2, \dots, n$
B_n	$1, 3, 5, \dots, 2n-1$
D_n	$1, 3, 5, \dots, 2n-1, n-1$
E_6	$1, 4, 5, 7, 8, 11$
E_7	$1, 5, 7, 9, 11, 13, 17$
E_8	$1, 7, 11, 13, 17, 19, 23, 29$
F_4	$1, 5, 7, 11$
G_2	$1, 5$
H_3	$1, 5, 9$
H_4	$1, 11, 19, 29$
$I_2(m)$	$1, m-1$

Poincaré Polynomials

Theorem (Chevalley '55, Solomon '66, Steinberg '68)

$$\sum_{w \in W} q^{\ell_S(w)} = \prod_{i=1}^{n} \frac{1 - q^{m_i+1}}{1-q}$$

Theorem (Shephard & Todd '54, Solomon '63)

$$\sum_{w \in W} q^{\ell_T(w)} = \prod_{i=1}^{n} (1 + m_i q)$$

Cayley Graphs

- W a finite group
- $R \subseteq W$ a symmetric subset of generators without identity:

$$\{r \in W : r \in R \Rightarrow r^{-1} \in R, r \neq 1\}$$

- $\Gamma(W, R)$ a Cayley graph:

$$V(\Gamma) = W \qquad E(\Gamma) = \{\{w, wr\} : r \in R\}$$

- A Cayley graph $\Gamma(W, R)$ is normal if R is closed under conjugation

Adjacency Spectrum

- Let $u, v \in V(\Gamma)$.
- The adjacency matrix: $A(u,v) = 1$ if $\{u,v\} \in E(\Gamma)$ and 0 otherwise.
- The adjacency spectrum is the set of eigenvalues of A
- Some known adjacency spectra:

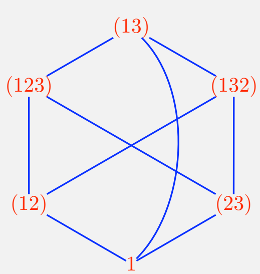
$\Gamma(\mathfrak{S}_n,$ all transpositions$)$ $(= \Gamma(\mathfrak{S}_n, T))$	Diaconis & Shahshahani, '81
$\Gamma(\mathfrak{S}_n,$ star transpositions$)$	Flatto, Odlyzko, and Wales, '85
$\Gamma(\mathfrak{S}_n,$ adjacent transpositions$)$ $(= \Gamma(\mathfrak{S}_n, S))$	Bacher, '94 (part)
$\Gamma(\mathfrak{S}_n,$ derangements$)$	—, '07

Distance Spectrum

- Let $u, v \in V(\Gamma)$.
- The distance $d(u,v) =$ the length of a shortest path from u to v
- The distance polynomial is the characteristic polynomial of d considered as a matrix
- The distance spectrum is the set of eigenvalues of d, *i.e.*, roots of the distance polynomial

Absolute Order Graph

- Absolute Order Graph on W: $\Gamma(W, T)$ where $T = \{$all reflections$\}$. Normal
- Example:



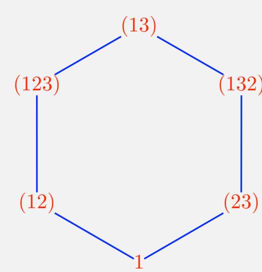
Absolute Order Graph $\Gamma(\mathfrak{S}_3, T)$

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & (12) & (123) & (13) & (132) & (23) \end{array} \\ \begin{array}{c} 1 \\ (12) \\ (123) \\ (13) \\ (132) \\ (23) \end{array} & \begin{pmatrix} 0 & 1 & 2 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 2 & 1 & 0 \end{pmatrix} \end{array}$$

Distance Matrix d

Weak Order Graph

- Weak Order Graph on W: $\Gamma(W, S)$ where $S = \{$simple reflections$\}$. Not normal
- Example:



Weak Order Graph $\Gamma(\mathfrak{S}_3, S)$

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & (12) & (123) & (13) & (132) & (23) \end{array} \\ \begin{array}{c} 1 \\ (12) \\ (123) \\ (13) \\ (132) \\ (23) \end{array} & \begin{pmatrix} 0 & 1 & 2 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 3 & 2 & 1 & 0 & 1 \\ 1 & 2 & 3 & 2 & 1 & 0 \end{pmatrix} \end{array}$$

Distance Matrix d

Distance Polynomials of Some Absolute Order Graphs

Type	Distance Polynomial
A_2	$(z-7)(z-1)(z+2)^4$
A_3	$(z-46)(z-2)^9(z+2)^5(z+6)^9$
A_4	$(z-326)(z-6)^{37}(z-2)^{25}(z+4)^{16}(z+6)^{25}(z+24)^{16}$
B_3	$z^9(z-100)(z-4)^{14}(z+4)^9(z+8)^{15}$
H_3	$(z-268)(z-8)^{18}(z-4)^{25}(z+2)^{32}(z+8)^{25}(z+12)^{18}(z+32)$
D_4	$z^{27}(z-544)(z-16)(z-8)^{36}(z-4)^{64}(z+8)^{20}(z+16)^{27}(z+32)^{16}$
F_4	$z^{160}(z-3600)(z-240)(z-48)^2(z-32)^{36}(z-24)^{16}(z-16)^{162}(z-12)^{256}(z-8)^{144}(z+16)^{117}(z+24)^{128}(z+48)^{105}(z+96)^{24}$
$I_2(5)$	$(z-13)(z-3)(z+2)^8$
$I_2(6)$	$(z-16)(z-4)(z+2)^{10}$
$I_2(7)$	$(z-19)(z-5)(z+2)^{12}$

Generalized Poincaré Polynomials and the Distance Spectrum

For each $\chi \in \mathrm{Irr}(W)$ introduce generalized Poincaré polynomial:

$$P_\chi(q) := \frac{1}{\chi(1)} \sum_{w \in W} \chi(w) q^{\ell_T(w)}$$

Theorem
The eigenvalues of the absolute order graphs $\Gamma(W,T)$ are given by

$$\eta_\chi = \left.\frac{dP_\chi(q)}{dq}\right|_{q=1},$$

with multiplicity $\chi(1)^2$ for each $\chi \in \mathrm{Irr}(W)$. Moreover, $\eta_\chi \in \mathbb{Z}$.

Proof Sketch

Consider

$$\mathcal{L} := \sum_{w \in W} \ell_T(w) w \in \mathbb{C}W.$$

Let ρ be left regular representation of W extended to $\mathbb{C}W$. Then $\mathcal{L}$ is represented by distance matrix:

$$\begin{aligned}\rho(\mathcal{L}_T)v &= \sum_w \ell_T(w)\rho(w)v = \sum_w \ell_T(w)wv \\ &= \sum_u \ell_T(uv^{-1})u = \sum_u d(u,v)u.\end{aligned}$$

(Last equality follows from bi-invariance of d.)

Proof Sketch—cont.

ℓ_T is a conjugacy class invariant, so

$$\mathcal{L}_T = \sum_K \ell_T(w_K) \sum_{w \in K} w = \sum_K \ell_T(w_K)\mathcal{I}_K,$$

where $\mathcal{I}_K$ is (up to constant) the trivial idempotent on class K, and hence constant on irreducible modules, by Schur's lemma. The irreducible modules appear with multiplicity equal to their dimensions, so taking traces yields the eigenvalues

$$\eta_\chi = \frac{1}{|\chi(1)|} \sum_K |K| \ell_T(w_K)\chi(w_K) = \left.\frac{dP_\chi(q)}{dq}\right|_{q=1},$$

each with multiplicity $\chi(1)^2$. Integrality follows by a theorem of Isaacs. □

Leading (?) Eigenvalue of Absolute Order Graphs

$P_1(q) =$ ordinary Poincaré polynomial $= \prod_i (1 + m_i q)$

Corollary

$$\eta_1 = |W| \sum_i \frac{m_i}{1+m_i}.$$

Conjecture

η_1 is the largest eigenvalue of the distance matrix of the absolute order graphs.

Generalized Poincaré Polynomial in Type A_{n-1}

- Can obtain remaining eigenvalues without knowing characters explicitly

Theorem (Molchanov '82, Stanley EC2)

The generalized Poincaré polynomial for the symmetric group $\mathfrak{S}_n = W(A_{n-1})$ is

$$P_\lambda(q) = \prod_{u \in \lambda} (1 + qc(u)),$$

where $\lambda \vdash n$ is a partition of n, and $c(u)$ is the content of box u (namely $j - i$ if box u is at (i, j) in Ferrers diagram of partition).

- Multiplicity of η_λ is f_λ^2, where $f_\lambda = n!/H_\lambda$ and H_λ is the product of all the hook lengths of λ.

Generalized Poincaré Polynomials in Other Types

Theorem (Molchanov '82)

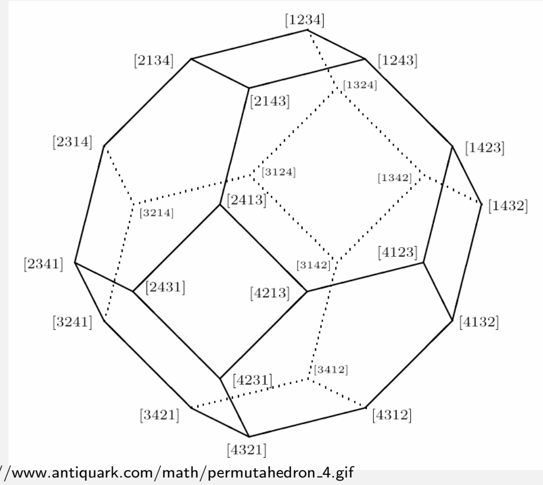
The generalized Poincaré polynomial for the hyperoctachedral group $W(B_n)$ is

$$P_{(\lambda,\mu)} = \prod_{u \in \lambda} [1 + q(c(u) + 1)] \prod_{v \in \mu} [1 + q(c(v) - 1)],$$

where $\lambda \vdash k$, $\mu \vdash m$, and $n = k + m$.

- Multiplicity of $\eta_{(\lambda,\mu)}$ is $f_{\lambda\mu}^2$, where
$$f_{\lambda\mu} = \frac{n!}{k!m!} f_\lambda f_\mu.$$
- Molchanov also computed generalized Poincaré polynomials in types D_n and $I_2(n)$

$\Gamma(\mathfrak{S}_4, S)$: The Permutahedron in $\mathbb{R}^3$



Source: http://www.antiquark.com/math/permutahedron_4.gif

Distance Polynomials of Some Weak Order Graphs

Type	Distance Polynomial
A_2	$z^2(z-9)(z+1)(z+4)^2$
A_3	$z^{17}(z-72)(z+4)^3(z+20)^3$
A_4	$z^{109}(z-600)(z+20)^6(z+120)^4$
B_3	$z^{38}(z-216)(z+8)^3(z^2+64z+384)^3$
D_4	$z^{179}(z-1152)(z+224)^4(z+32)^8$
D_5	$z^{1899}(z-19200)(z+2880)^5(z+320)^{15}$
H_3	$z^{104}(z-900)(z^2+248z+3856)^3(z+24)^4(z+12)^5$
F_4	$z^{1127}(z-13824)(z+192)^{16}(z^2+2688z+313344)^4$
E_6	$z^{51803}(z-933120)(z+112320)^6(z+8640)^{30}$
$I_2(5)$	$z^4(z-25)(z+1)(z^2+12z+16)^2$
$I_2(6)$	$z^5(z-36)(z+2)^2(z^2+16z+16)^2$
$I_2(7)$	$z^6(z-49)(z+1)(z^3+24z^2+80z+64)^2$

The Distance Spectra of Cayley Graphs of Coxeter Groups
└Distance Spectra of Weak Order Graphs

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Observations

- $\Gamma(W,S)$ not normal, so no simple formula involving characters
- Lots of zeros (huge degeneracy) and very few distinct eigenvalues!
- Spectrum not integral in general, but ...

Conjecture

The distance spectra of the weak order graphs of types A, D, and E each contain only four distinct integral eigenvalues.

- Have partial proof of conjecture and explanations for *some* of these observations

A Variant of the Distance Matrix

- Difficulty is that distance matrix is too large: size $|W| \times |W|$
- Instead of working with group elements, can work with roots
- Use Coxeter combinatorics to find a smaller matrix with size $|\Phi| \times |\Phi|$ and related spectrum

The Angle Operator

- Let $\Psi = \{|\alpha\rangle : \alpha \in \Phi\}$ be the permutation representation of W
- Define angle operator Θ by

$$\Theta |\beta\rangle = \sum_{\alpha} \theta_{\alpha\beta} |\alpha\rangle ,$$

where $\theta_{\alpha\beta}$ is the angle between α and β.

Theorem

The distance spectrum of the weak order graphs can be recovered from the spectrum of Θ.

Proof Idea

Let Π be set of positive roots (all $c_i \geq 0$). Well known that

$$\ell_S(w) = |w\Pi \cap (-\Pi)|,$$

so

$$d_S(u,v) = \ell_S(u^{-1}v) = |v\Pi \cap u(-\Pi)|.$$

Now define $\widetilde{d}_S(u,v) = |u\Pi \cap v\Pi|$. Can show

$$\text{Spec}\, d_S = \{\lambda_1, \lambda_2, \ldots, \lambda_{|W|}\} \Leftrightarrow \text{Spec}\, \widetilde{d}_S = \{\lambda_1, -\lambda_2, \ldots, -\lambda_{|W|}\}.$$

Proof Idea—cont.

Define $|\psi_w\rangle := \sum_{\alpha>0} |w\alpha\rangle$. It follows that

$$\widetilde{d}_S(u,v) = \langle \psi_u | \psi_v \rangle .$$

This has same spectrum as

$$\mathfrak{D} = \sum_{w} |\psi_w\rangle \langle \psi_w| .$$

Can show that

$$\mathfrak{D}_{\alpha\beta} = |\{w \in W : w\alpha > 0 \text{ and } w\beta > 0\}| .$$

Proof Idea—cont.

Show for dihedral groups that

$$\mathfrak{D} = \frac{|W|}{2\pi}(\pi - \Theta),$$

then show it holds for all other reflection groups by considering cosets of dihedral subgroups. □

Cyclic Eigenvectors in Types A, D, and E

Theorem

Let $(\alpha, \beta, \gamma) \subset \Phi$ be distinct, coplanar roots with $\alpha + \beta + \gamma = 0$. Define cyclic subspace $\Psi_c \subset \Psi$ to be space spanned by vectors of form

$$|\psi_{\alpha\beta\gamma}\rangle := \Big(|\alpha\rangle + |\beta\rangle + |\gamma\rangle \Big) - \Big(|-\alpha\rangle + |-\beta\rangle + |-\gamma\rangle \Big).$$

Let W be of type A, D, or E. Then

$$\Theta \, |\psi_{\alpha\beta\gamma}\rangle = \frac{2\pi}{3} \, |\psi_{\alpha\beta\gamma}\rangle \, .$$

A Combinatorial Problem

Problem

What is the dimension of Ψ_c?

- In type A_{n-1} roots are of form $e_i - e_j$ for $1 \le i < j \le n$.
- Can map roots to directed edges in complete graph K_n.
- Ψ_c can be identified with cycle space of K_n, which has dimension $\binom{n-1}{2}$.
- In type D can show $\dim \Psi_c = n(n-2)$. (Proof by representation theory.) Combinatorial argument desired.

Distance Polynomial of Weak Order Graph in Type A

Theorem

The distance polynomial of $\Gamma(\mathfrak{S}_n, S)$ is

$$q^{n! - (1/2)(n^2 - n + 2)} \left(q - \frac{n!}{2} \binom{n}{2} \right) \times \left(q + \frac{n!}{6} \right)^{(n-1)(n-2)/2} \left(q + \frac{(n+1)!}{6} \right)^{n-1} .$$

Proof.

By constructing remaining eigenvectors. □

Things To Do

- Distance polynomial in type D obtained similarly. (Type I also known.) What about the other types?
- Applications? Chemistry?
- Can similar methods help to determine the adjacency spectrum of weak order graphs?