Some Reflection Group Numerology

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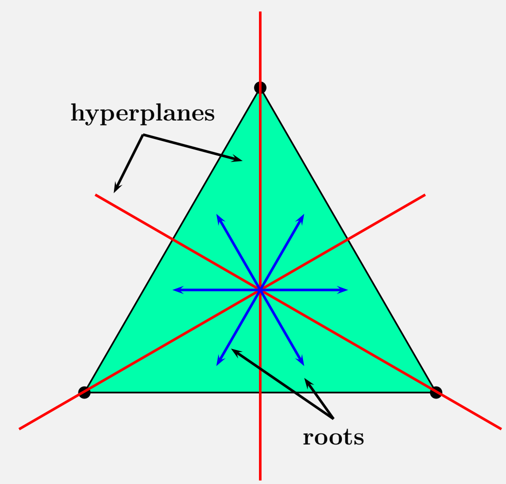
Outline

Root Systems and Reflection Groups

- V an n dimensional inner product space over $\mathbb{R}$
- Given $\alpha \in V$, reflection $t_\alpha : V \to V$ fixes hyperplane $H_\alpha := \{v \in V | (\alpha, v) = 0\}$ (pointwise) and sends α to $-\alpha$.
- $\Phi \subset V$ is a root system if
  - $t_\alpha \Phi = \Phi$, $\alpha \in \Phi$, and
  - $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\}$ for all $\alpha \in \Phi$.
- Let $W(\Phi)$ be the group generated by reflections t_α, $\alpha \in \Phi$
- If W is finite it is called a finite reflection group or finite Coxeter group
- W is a Weyl group if Φ is crystallographic: $2(\alpha, \beta)/(\beta, \beta) \in \mathbb{Z}$

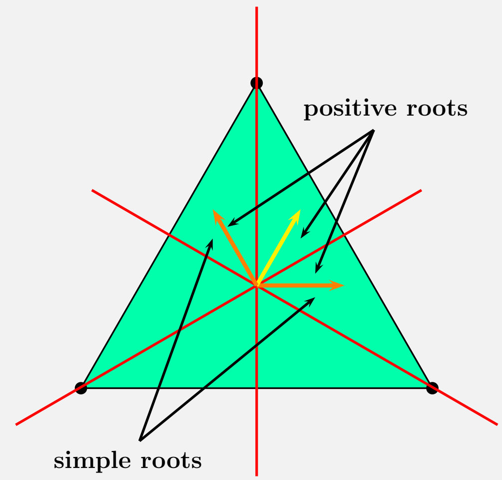
Example: Symmetries of a triangle



Simple Systems

- $\Delta \subset \Phi$ is a simple system provided
  - Δ is a basis for V.
  - Every root $\alpha \in \Phi$ can be written $\alpha = \sum_{\alpha_i \in \Delta} c_i \alpha_i$ where all $c_i \geq 0$ or all $c_i \leq 0$.
- The elements of a simple system are called simple roots.
- The positive roots Φ^+ are those for which $c_i > 0$ for all i.
- In crystallographic case, $c_i \in \mathbb{Z}$ for all i.

Positive Roots and Simple Roots



Generators and Relations

- Fix a simple system $\Delta \in \Phi$.
- Write s_α instead of t_α whenever $\alpha \in \Delta$. (The simple reflections.)

Theorem (The Coxeter Presentation)

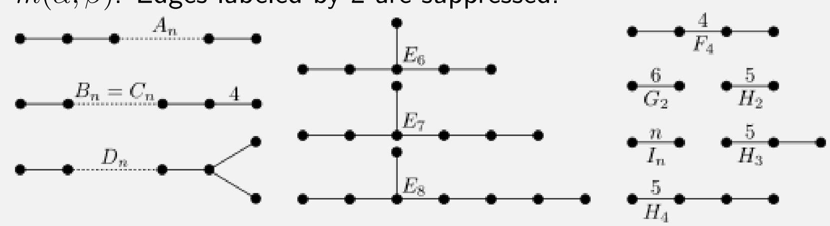
W is generated by the set of simple reflections subject only to the relations

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1 \ (\alpha, \beta \in \Delta).$$

- The minimum length of a word w written in terms of simple (respectively, all) reflections is called the length (respectively, absolute length) of w.

The Classification

The Coxeter graph of W is the graph with one vertex for each simple reflection and with edges (s_α, s_β) labeled by the integers $m(\alpha, \beta)$. Edges labeled by 2 are suppressed.



W is irreducible if the Coxeter graph is connected. All Coxeter groups are direct products of irreducible ones.

Coxeter Transformations

- The Coxeter element of W is $c = \prod_{s \in S} s$ (unique up to conjugacy)
- The order of c is h, the Coxeter number
- The eigenvalues of c (in the reflection representation) are of the form ζ^{m_i} for some primitive h^{th} root of unity ζ.
- The numbers $m_1, m_2, \ldots m_n$ are the exponents of W

The Exponents of Reflection Groups

Type	$m_1, m_2, \ldots, m_n$
A_n	$1, 2, \ldots, n$
B_n	$1, 3, 5, \ldots, 2n-1$
D_n	$1, 3, 5, \ldots, 2n-1, n-1$
E_6	$1, 4, 5, 7, 8, 11$
E_7	$1, 5, 7, 9, 11, 13, 17$
E_8	$1, 7, 11, 13, 17, 19, 23, 29$
F_4	$1, 5, 7, 11$
G_2	$1, 5$
H_3	$1, 5, 9$
H_4	$1, 11, 19, 29$
$I_2(m)$	$1, m-1$

Exponents in Surprising Places

Theorem (Chevalley '55, Solomon '66, Steinberg '68)

Let $\ell_S(w)$ denote the minimum length of w as a word in the simple reflections. Then

$$\sum_{w \in W} q^{\ell_S(w)} = \prod_{i=1}^{n} \frac{1 - q^{m_i+1}}{1-q}$$

Theorem (Shephard & Todd '54, Solomon '63)

Let $\ell_T(w)$ denote the minimum length of w as a word in all the reflections. Then

$$\sum_{w \in W} q^{\ell_T(w)} = \prod_{i=1}^{n} (1 + m_i q)$$

Exponents in Surprising Places—cont.

Theorem (Brieskorn '71)

Let $M = V_{\mathbb{C}} \backslash \{\bigcup H_\alpha\}$. Then

$$\sum_{i \geq 0} \dim\, H^i(M, \mathbb{C}) t^i = \prod_{i=1}^{n} (1 + m_i t)$$

And the list goes on.... Exponents appear in many beautiful formulas for nonnesting partitions, noncrossing partitions, cluster algebras, etc.

Main Question

Shephard and Todd proved their result case-by-case for unitary reflection groups (a generalization of real reflection groups). The other proofs all used invariant theory. (Brieskorn reduced his problem to the Shephard-Todd result.)

Can one find a simple combinatorial explanation for the exponents?

Normal Forms for Reflection Group Elements

- Look again at the Shephard-Todd formula

$$\sum_{w \in W} q^{\ell_T(w)} = \prod_{i=1}^{n} (1 + m_i q).$$

- Wouldn't it be nice if this were the shadow of something deeper?

Theorem (Known?)

Let W be of type A_n or B_n and let T be the set of all reflections of W. Then there exists a partition of T into classes T_i with $|T_i| = m_i$ satisfying

$$\sum_{w \in W} q^{\ell_T(w)} w = \prod_{i=1}^{n} (1 + qX_i)$$

as an identity in the group algebra $\mathbb{R}[q]W$, where $X_i := \sum_{t \in T_i} t$.

Proof

Normal Forms—cont.

- But the theorem seems to fail for all other types! Why?
- Something funny already for type D_n. To see what it is, we need to take a little detour...

Hyperplane Arrangements and the Lattice of Flats

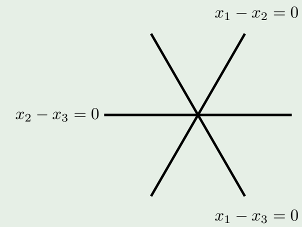
- A collection of hyperplanes is called a hyperplane arrangement.
- The intersection of any subcollection of hyperplanes is called a flat.
- To every arrangement $\mathcal{A}$ we associate the intersection poset whose elements are the flats, ordered by reverse inclusion so that $A \leq B \Leftrightarrow A \supseteq B$.
- The arrangement is central if all the hyperplanes intersect in a point, in which case the intersection poset is a lattice (joins and meets exist), called the lattice of flats.

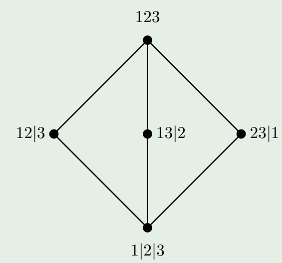
The W-Partition Lattice

- A reflection arrangement is the set of all reflecting hyperplanes of a reflection group W.
- The lattice of flats of a reflection arrangement L_W is called the W-partition lattice.

The $W(\mathsf{A}_2)$ Partition Lattice

Example





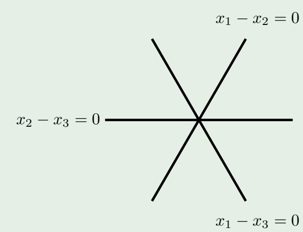
The Möbius Function and the Characteristic Polynomial

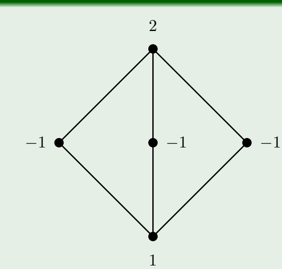
- The Möbius function $\mu(x, y)$ is defined recursively. We only need $\mu(\hat{0}, x)$, which is computed as follows.
  1. $\mu(x, x) = 1$.
  2. $\mu(\hat{0}, x) = -\sum_{y<x} \mu(\hat{0}, y)$
- The characteristic polynomial of L is

$$\chi(\mathcal{A}, q) = \sum_{x \in L} \mu(\hat{0}, x) q^{\dim(x)}.$$

The $W(\mathsf{A}_2)$ Partition Lattice–cont.

Example





$$\chi(\mathcal{A}, q) = \sum_{x \in L} \mu(\hat{0}, x)\, q^{\dim x} = q^2 - 3q + 2$$

Characteristic Polynomials of Reflection Arrangements

Why is all this relevant?

Theorem (Orlik and Solomon '80)

Let L_W be the partition lattice associated to the reflection group W. Then

$$\chi(L_W, q) = \prod_{i=1}^{n}(q - m_i),$$

where the m_i are the exponents of W.

But how to compute $\chi(L_W, q)$?

Finite Field Method

- Recall that the defining equation of a hyperplane can be written $a_1x_1 + \cdots a_nx_n = b$ for some real numbers $\{a_1, a_2, \ldots, a_n, b\}$.
- In many cases of interest the numbers $a_1, a_2, \ldots, a_n, b$ are integers (an integral arrangement).
- When this holds there is a particularly nice way to compute the characteristic polynomial.
- For any positive integer q let $\mathcal{A}_q$ denote the hyperplane arrangement $\mathcal{A}$ with defining equations reduced mod q.

The Characteristic Polynomial for Integral Arrangements

Theorem (Crapo and Rota '71, Orlik and Terao '92, Blass and Sagan '96, Athanasiadis '96, Björner and Ekedahl '96)

For sufficiently large primes q,

$$\chi(\mathcal{A}, q) = \#\left(\mathbb{F}_q^n - \bigcup_{H \in \mathcal{A}_q} H\right),$$

where $\mathbb{F}_q^n$ denotes the vector space of dimension n over the finite field with q elements.

Remarks

- We need large q to avoid lowering the dimension of any of the flats by accident.
- But two polynomials that agree for enough values of q are equal.
- Identifying $\mathbb{F}_q^n$ with $\{0, 1, \ldots, q-1\}^n = [0, q-1]^n$, $\chi(\mathcal{A}, q)$ is the number of points in $[0, q-1]^n$ that do *not* satisfy modulo q the defining equations of any of the hyperplanes in $\mathcal{A}$.

Weyl Arrangements

- Weyl arrangements are integral, so method applies.
- Three infinite families associated to types A_{n-1}, D_n, and B_n, respectively:

$$\begin{aligned} \mathcal{A}_{n-1} &= \{x_i - x_j = 0 | 1 \le i \le j \le n\} \\ \mathcal{D}_n &= \mathcal{A}_n \cup \{x_i + x_j = 0 | 1 \le i \le j \le n\} \\ \mathcal{B}_n &= \mathcal{D}_n \cup \{x_i = 0 | 1 \le i \le n\} \end{aligned}$$

Computing the Characteristic Polynomial I

- What is $\chi(\mathcal{A}_{n-1})$?
- According to the finite field method, we want the number of points in $[0, q-1]^n$ satisfying $x_i \neq x_j$ for all $1 \le i \le j \le n$.
- This is the same thing as asking for vectors $(x_1, x_2, \ldots, x_n)$ all of whose entries are distinct mod q.
- Well, we can pick x_1 in q ways, then x_2 in $q-1$ ways, and so on. Thus

$$\chi(\mathcal{A}_{n-1}, q) = q(q-1)(q-2)\cdots(q-n+1).$$

Computing the Characteristic Polynomial II

- What is $\chi(\mathcal{B}_n)$?
- Now we want to count the points satisfying $x_i \neq x_j$, $x_i \neq -x_j$, and $x_i \neq 0$.
- Since we do not allow 0, there are only $q-1$ (nonzero) choices for the first entry, $q-3$ nonzero choices for the second entry (because we must avoid the first entry and its negative), *etc.*.
- Thus

$$\chi(\mathcal{B}_n, q) = (q-1)(q-3)\cdots(q-2n+1).$$

Computing the Characteristic Polynomial III

- What is $\chi(\mathcal{D}_n)$?
- Now we want to count the points satisfying $x_i \neq x_j$, and $x_i \neq -x_j$, but this time we allow 0 as an entry.
- There are two possibilities. If zero is not present, then $\mathcal{B}_n$ case. If zero is present, then $\mathcal{B}_{n-1}$ case for remaining entries.

III–continued

- There are n choices for the placement of the zero, so we get

$$\begin{aligned}\chi(\mathcal{D}_n, q) &= (q-1)(q-3)\cdots(q-2n+1)\\ &\quad + (q-1)(q-3)\cdots(q-2n+3)(n)\\ &= (q-1)(q-3)\cdots(q-2n+3)(q-n+1)\end{aligned}$$

- Basic reason for failure of normal form theorem in type D_n (as well as other types).
- Is there another way?
- Instead of $\mathbb{F}_q^n$ use a symmetry adapted lattice.

Reflection/Coxeter Groups
Reflection Arrangements
Finite Field Method

Ehrhart Quasipolynomials

- Let $\mathbb{N} := \{0, 1, \ldots\}$.
- A function $f : \mathbb{N} \to \mathbb{R}$ is a quasipolynomial with quasiperiod M if there exist polynomials $f_1, \ldots, f_M : \mathbb{N} \to \mathbb{R}$ such that $f(m) = f_i(m)$ for all $m \in \mathbb{N}$ satisfying $m = i \mod M$.
- A rational polytope P is one whose vertices have rational coordinates.
- For any integer k the Ehrhart quasipolynomial $\iota(P, k) := |kP \cap \mathbb{Z}|$ counts lattice points in dilate kP.
- Similarly define $\bar{\iota}(P, k) := |kP^0 \cap \mathbb{Z}|$, where P^0 is the interior of P.

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Lattice Point Counting Formula

- Let Φ be crystallographic with simple roots $\{\alpha_i\}$.
- Recall $c_i(\alpha)$ given by $\alpha = \sum_i c_i(\alpha)\alpha_i$.
- There exists a unique highest root $\widetilde{\alpha}$ with largest coefficients, denoted $\widetilde{c}_i$.

Theorem (Haiman '93, Blass and Sagan '96, Athanasiadis '96, Terao et al '07)

Let $\mathcal{A}$ be the Weyl arrangement associated to Φ, and let σ be the rational simplex in the nonnegative orthant of $\mathbb{Z}^n$ bounded by the hyperplane $\sum_i \widetilde{c}_i x_i = 1$. Let $M := \operatorname{lcm}(\widetilde{c}_1, \ldots, \widetilde{c}_n)$. Then for q relatively prime to M, $\chi(\mathcal{A}, q) = C\bar{\iota}(\sigma, q)$, where $C := n! \prod_i \widetilde{c}_i$.

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Exponents from Counting

Corollary

Let W be a Weyl group with Coxeter number h, reflection arrangement $\mathcal{A}$, and minimum quasiperiod M. Then the characteristic polynomial $\chi(\mathcal{A}, q)$ vanishes (and therefore q is an exponent) whenever $q \leq h-1$ and $(q, M) = 1$.

Proof (Was already known that q is an exponent if $q \leq h-1$ is coprime to h.)

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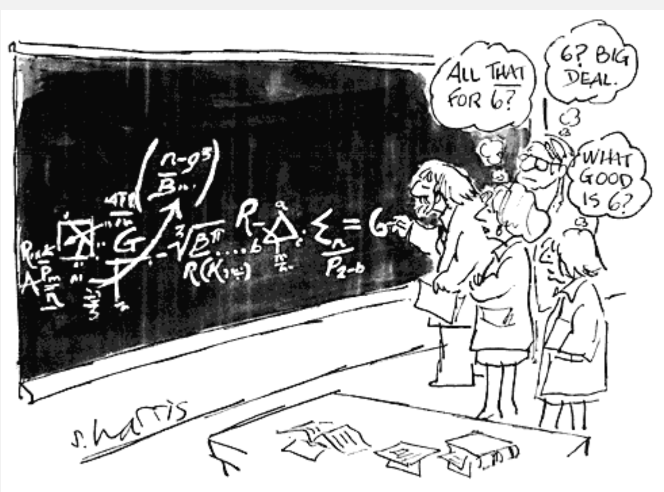
Exponents from Counting–cont.

Type	Coefficients of $\widetilde{\alpha}$	M	h	zeros of χ from cor.	missing
A_n	1,1,...,1	1	n+1	1,2,...,n	
B_n	1,2,2,...,2	2	2n	1,3,5,...,2n-1	
D_n	1,1,1,2,...,2	2	2n-2	1,3,5,...,2n-3	n-1
E_6	1,1,2,2,2,3	6	12	1,5,7,11	4,8
E_7	1,2,2,2,3,3,4	12	18	1,5,7,11,13,17	9
E_8	2,2,3,3,4,4,5,6	60	30	1,7,11,13,17,19,23,29	
F_4	2,2,3,4	12	12	1,5,7,11	
G_2	2,3	6	6	1,5	

Work in Progress

- Can get rest of exponents from generating function, but not pretty.
- How is this related to theorem of Shapiro (?), Kostant ('59), Macdonald ('72) concerning root height partitions?
- Is there a counting interpretation of the exponents in the noncrystallographic case?

The End



Normal Forms–cont.

Proof.

Basically one shows that one can choose the given reflections as minimal absolute length coset representatives for a maximal chain of (parabolic) subgroups. For example, for type A_{n-1} (listing the reflections by the corresponding root and letting $T_1 = \emptyset$ for convenience)

$$T_2 := \{e_2 - e_1\}$$
$$T_3 := \{e_3 - e_2, e_3 - e_1\}$$
$$T_4 := \{e_4 - e_3, e_4 - e_2, e_4 - e_1\}$$
$$\vdots$$
$$T_n := \{e_n - e_{n-1}, e_n - e_{n-2}, \ldots, e_n - e_1\}$$

□

Back to main.

Roots of the Characteristic Polynomial

- $\bar{\iota}(s,q)$ counts all x's with $x_i > 0$ and $\sum_i \widetilde{c}_i x_i < q$.
- The smallest possible point in interior of σ is $(1, 1, \ldots, 1)$.
- $\sum_i \widetilde{c}_i = h - 1$ (h the Coxeter number).
- Therefore $\bar{\iota}(s,q) = 0$ for $q \le h - 1$.
- But $\chi(\mathcal{A}, q) = C\bar{\iota}(\sigma, q)$ for q prime to the quasiperiod.
- So $\chi(\mathcal{A}, q) = 0$ for $q \le h - 1$ and $(q, M) = 1$.

Return