Sylow permutation characters in the symmetric group

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Abstract

An algorithm is developed for computing the exact multiplicities of the irreducible characters of $\mathfrak{S}_n$ appearing in the permutation character $1_Q^{\mathfrak{S}_n}$, where Q is a Sylow-q-subgroup of $\mathfrak{S}_n$. As a bonus, the algorithm provides a method for calculating the values $1_Q^{\mathfrak{S}_n}(\nu)$ at any partition ν.

Keywords: Symmetric group, Sylow-p-subgroup, cycle index, wreath product, permutation character, irreducible characters

1 Introduction

For the last several years, Eugenio Giannelli, Stacey Law, and their colleagues have investigated the properties of the induced characters $\theta^{\mathfrak{S}_n}$, where θ is an irreducible character of a Sylow-q-subgroup of the symmetric group $\mathfrak{S}_n$ (see, *e.g.*, [3] and references therein). In [2] the authors demonstrate that, with a few exceptions, every irreducible character χ^λ of $\mathfrak{S}_n$ (where λ is a partition of n) appears as a constituent of the permutation character $1_Q^{\mathfrak{S}_n}$. In that same paper they ask for a combinatorial description of the multiplicities that occur. In addition, in ?? the authors ask for a uniform method to evaluate these characters at specific partitions.

In this paper we supply an algebraic (rather than combinatorial *per se*) answer to both questions in the form of an algorithm. The algorithm derives from an analysis of the cycle indices of the Sylow-q-subgroups of $\mathfrak{S}_n$ carried out in [7], together with some results from symmetric function theory. Briefly, the main result is as follows. [1]

Every $\pi \in \mathfrak{S}_n$ can be written as a product of disjoint cycles. The type vector of π is the vector $\boldsymbol{c} := (c_1(\pi), c_2(\pi), \ldots, c_n(\pi))$, where $c_i(\pi)$ is the number of cycles of π of length i. As $\sum_{i=1}^{n} ic_i(\pi) = n$, each type vector $\boldsymbol{c}$ corresponds to an integer partition ν of n (written $\nu \vdash n$), namely $1^{c_1(\pi)}2^{c_2(\pi)} \cdots n^{c_n(\pi)}$ (in multiplicity notation). The partition ν obtained in this way is the cycle type of π, and is denoted $\rho(\pi)$. It is easy to see that the cycle type is preserved under conjugation, so we may write C_ν for the conjugacy class of all partitions having cycle type ν. It is well-known (*e.g.*, [10], Prop. 1.3.2) that if $\nu = 1^{c_1} \cdots n^{c_n}$ then the cardinality of C_ν equals $n!/z_\nu$, where

$$z_\nu := 1^{c_1} c_1! \cdots n^{c_n} c_n! \tag{1}$$

is the cardinality of the centralizer of any permutation π with $\rho(\pi) = \nu$. If ψ is a class function on $\mathfrak{S}_n$ we write $\psi(\rho(\pi))$ for $\psi(\pi)$.

Let χ^λ denote the irreducible character of $\mathfrak{S}_n$ associated to the partition λ. Write

$$(\psi, \varphi)_G = \frac{1}{|G|} \sum_{g \in G} \overline{\psi(g)} \varphi(g)$$

for the usual character theoretic inner product; then $(\chi^\lambda, 1_Q^{\mathfrak{S}_n})_{\mathfrak{S}_n}$ is the multiplicity of χ^λ in $1_Q^{\mathfrak{S}_n}$. Let $Z_Q(t_1, t_2, \ldots, t_n)$ be the cycle index of Q (considered as a permutation subgroup of $\mathfrak{S}_n$), where the t_i's are algebraically independent indeterminates. Let $p_n := \sum_i x_i^n$ be the power sum symmetric function (in any number of variables), and for any partition $\nu = (\nu_1, \nu_2, \ldots)$, set $p_\nu := \prod_i p_{\nu_i}$. Write $Z_Q(p)$ for $Z_Q(p_1, p_2, \ldots, p_n)$ and z_μ for the cardinality of the centralizer in $\mathfrak{S}_n$ of a permutation having cycle type μ. Also, the Kronecker delta is

$$\delta_{\mu\nu} = \begin{cases} 1, & \text{if } \mu = \nu, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1. *The multiplicities $(\chi^\lambda, 1_Q^{\mathfrak{S}_n})$ may be obtained from by replacing each instance of p_μ in $Z_Q(p)$ by $\chi^\lambda(\mu)$. The values $1_Q^{\mathfrak{S}_n}(\nu)$ may be obtained by replacing each instance of p_μ in $Z_Q(p)$ by $z_\nu \delta_{\nu\mu}$.*

[1] For background material on the symmetric group, symmetric functions, and representation theory of the symmetric group, see [4], [6], [9], and [11].

In the remainder of this paper we prove Theorem 1 and give several examples of its use.

2 Preliminaries

In this section we collect some results that will be needed for the proof of Theorem 1.

2.1 Permutation characters

We begin with a well-known basic result.

Lemma 1. *Let G be a finite group acting transitively on a finite set X. Let $C(g)$ denote the conjugacy class of $g \in G$, and let G_x be the stabilizer of $x \in X$. Then the number of fixed points of g on X is*

$$\operatorname{fix}(g) = \frac{|C(g) \cap G_x|}{|C(g)|}|X|.$$

Proof. ([1], Prop. 8.4) By double counting. Let

$$S := \{(c, x) : cx = x, c \in C(g), x \in X\}.$$

On the one hand, $|S| = |C(g)| \operatorname{fix}(c) = |C(g)| \operatorname{fix}(g)$, as fix is constant on conjugacy classes. On the other hand, $|S| = |X||C(g) \cap G_x|$. □

If H is a subgroup of a finite group G then G acts transitively on G/H, and the permutation character 1_H^G (the induction of H to G) counts the number left cosets fixed by G ([4], 4.8). Hence, by the lemma,

$$1_H^G(g) = \frac{|C(g) \cap H|}{|C(g)|} \frac{|G|}{|H|}. \tag{2}$$

If H is a subgroup of $\mathfrak{S}_n$, then

$$1_H^{\mathfrak{S}_n}(\nu) = \frac{|C_\nu \cap H|}{|C_\nu|} \frac{n!}{|H|} = z_\nu \frac{|C_\nu \cap H|}{|H|}. \tag{3}$$

2.2 The Frobenius characteristic and the Hall inner product

Given any class function ψ on $\mathfrak{S}_n$ we can define its *Frobenius characteristic* $\mathrm{ch}(\psi)$ by

$$\mathrm{ch}(\psi) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \psi(\pi) p_{\rho(\pi)} = \sum_{\nu \vdash n} z_\nu^{-1} \psi(\nu) p_\nu.$$

Using (3) we obtain

$$\mathrm{ch}(1_H^{\mathfrak{S}_n}) = \frac{1}{|H|} \sum_\nu |C_\nu \cap H| p_\nu = Z_H(p_1, p_2, \dots), \tag{4}$$

where Z_H is the *cycle index* (or *cycle indicator function*) of the group H, written in terms of the power sum symmetric functions. [2]

The set of power sum symmetric functions in n variables form a basis for the algebra Λ_n of all symmetric functions in n variables. The *Hall inner product*, defined by

$$\langle p_\mu, p_\nu \rangle_{\Lambda_n} = z_\mu \delta_{\mu\nu} \tag{5}$$

turns Λ_n into an inner product space. One of the key features of the Frobenius characteristic is that it is an isometry between the space of class functions on $\mathfrak{S}_n$ and the space Λ_n ([11], Proposition 7.18.1) [3]

$$\langle \psi, \chi \rangle_{\mathfrak{S}_n} = \langle \mathrm{ch}(\psi), \mathrm{ch}(\chi) \rangle_{\Lambda_n}.$$

The Hall inner product satisfies many important identities; we shall need the following two. The first one says that the Schur functions s_λ form an orthonormal basis for Λ_n:

$$(s_\lambda, s_\nu) = \delta_{\lambda,\nu} \tag{6}$$

The second identity (essentially, this is the Murnaghan-Nakayama rule; see ([11], Theorem 7.18.5)) relates the power sum symmetric functions, Schur functions, and irreducible characters χ^λ of $\mathfrak{S}_n$:

$$(p_\nu, s_\lambda) = \chi^\lambda(\nu). \tag{7}$$

We may use these formulae to expand the permutation character $1_H^{\mathfrak{S}_n}$ in terms of irreducible characters χ^λ. Specifically, if

$$1_H^{\mathfrak{S}_n} = \sum_\lambda a_\lambda \chi^\lambda \tag{8}$$

[2] See ([11], Sec. 7.24), which we follow here. According to Polya's theory, $Z_H(p)$ equals the so-called *pattern inventory* of H acting on the set of colorings of a set of size n, but we do not need this fact here.

[3] Henceforth we drop the subscript on the inner products, as it should be clear from context.

then

$$Z_H = \mathrm{ch}(1_H^{\mathfrak{S}_n}) = \sum_\lambda a_\lambda s_\lambda, \tag{9}$$

and conversely. We may therefore restate Theorem 1 as follows.

Theorem 1. *Let H be a subgroup of $\mathfrak{S}_n$, and let Z_H be its cycle index written in terms of the power sum symmetric functions. Then*

$$1_H^{\mathfrak{S}_n} = \sum_\lambda (Z_H, s_\lambda)\chi^\lambda. \tag{10}$$

Moreover, for any partition $\nu \vdash n$,

$$1_H^{\mathfrak{S}_n}(\nu) = (Z_H, p_\nu). \tag{11}$$

Proof. Equation (10) is immediate from (8) and (9), while (11) follows from (6), (7) and (10). □

3 Permutation characters of Sylow subgroups

In this section we illustrate Theorem 1 by applying it to Sylow subgroups of the symmetric group. For this, we need a bit more preparation.

3.1 Sylow-q-subgroups of $\mathfrak{S}_n$

Let q be a prime. [4] It is a classical result of Kalužnin that the Sylow-q-subgroups of $\mathfrak{S}_n$ may be obtained as cartesian products of iterated wreath products of cyclic groups. [5] Let $\mathbb{Z}_q$ be the cyclic subgroup of order q. Define the *iterated wreath product*

$$W_{m,q} := \underbrace{\mathbb{Z}_q \wr \mathbb{Z}_q \wr \cdots \wr \mathbb{Z}_q}_{m \text{ times}},$$

where $W_{m+1,q} := W_{m,q} \wr \mathbb{Z}_q$ and $W_{0,q} := 1$. The group $W_{m,q}$ is most simply viewed as the subgroup of the automorphism group of the complete q-ary tree of depth m generated by

[4] Unfortunately, we cannot use the letter 'p' here to denote a prime, as it is too easily confused with the standard notation for power sum symmetric functions.

[5] For more details about the ideas in this section, see [7] and references therein.

cyclic permutations of the nodes of the subtrees at each level. [6] As such a tree has q^m children, $W_{m,q}$ is naturally as subgroup of $\mathfrak{S}_{q^m}$. By a simple inductive argument, we see that $W_{m,q}$ has order $q^{\mu(m)}$, where

$$\mu(m) = q^{m-1} + q^{m-2} + \cdots + 1 = \frac{q^m - 1}{q-1}.$$

Given Sylow's theorems, the following result is not difficult (see, *e.g.*, [7]).

Theorem 2. *(Kalužnin, [5]) Let q be a prime.*

- *A Sylow-q-subgroup of $\mathfrak{S}_{q^m}$ is isomorphic to an iterated wreath product $W_{m,q}$.*
- *Let*

$$n = b_0 + b_1 q + \cdots + b_r q^r$$

be the q-adic expansion of n. Then a Sylow-q-subgroup of $\mathfrak{S}_n$ is isomorphic to the direct product

$$Q := W_{0,q}^{b_0} \times W_{1,q}^{b_1} \times W_{2,q}^{q_2} \times \cdots \times W_{r,q}^{b_r}. \tag{12}$$

3.2 Cycle indices of Sylow subgroups

It was shown in [7] that the cycle indices of Sylow subgroups of $\mathfrak{S}_n$ may be obtained using a recursive procedure, as follows. Fix a prime q, and define

$$A^{(r)}(k) := \frac{1}{q}([A^{(r-1)}(k)]^q + (q-1)A^{(r-1)}(k+1)) \quad \text{with} \quad A^{(0)}(k) := p_{q^k}. \tag{13}$$

Then

$$Z_{W_{r,q}} = A^{(r)}(0) \qquad (r \geq 1). \tag{14}$$

This yields, for instance,

$$\begin{aligned} Z_{W_{1,q}} = A^{(1)}(0) &= \frac{1}{q}\left([A^{(0)}(0)]^q + (q-1)A^{(0)}(1)\right) \\ &= \frac{1}{q}\left(p_1^q + (q-1)p_q\right) \\ &= \frac{1}{q}\left(p_{[1]^q} + (q-1)p_{[q]}\right) \end{aligned} \tag{15}$$

[6] An *r-tree* is just a tree consisting of one root with r children. The *complete r-ary tree of depth m* is the rooted tree constructed inductively by attaching r-trees to the leaves of the complete r-ary tree of depth $m-1$, where the complete r-ary tree of depth zero is just a single node.

and

$$
\begin{aligned}
Z_{W_{2,q}} = A^{(2)}(0) &= \frac{1}{q}\left([A^{(1)}(0)]^q + (q-1)A^{(1)}(1)\right) \\
&= \frac{1}{q^{q+1}}(p_1^q + (q-1)p_q)^q + \frac{q-1}{q^2}(p_q^q + (q-1)p_{q^2}) \\
&= \frac{1}{q^{q+1}}\sum_{k=0}^{q}\binom{q}{k} p_1^{qk}((q-1)p_q)^{q-k} + \frac{(q-1)}{q^2}(p_q^q + (q-1)p_{q^2}) \\
&= \frac{1}{q^{q+1}}\sum_{k=0}^{q}\binom{q}{k}(q-1)^{q-k}p_{[1]^{qk}[q]^{q-k}} + \frac{(q-1)}{q^2}(p_{[q]^q} + (q-1)p_{[q^2]}), \qquad (16)
\end{aligned}
$$

where for clarity we adopt the notational convention that a partition $\nu = 1^{i_1}2^{i_2}\cdots r^{i_r}$ is now written $\nu = [1]^{i_1}[2]^{i_2}\cdots[r]^{i_r}$. Finally, if

$$n = b_0 + b_1 q + b_2 q^2 + \cdots + b_r q^r$$

be the q-adic expansion of n, then

$$Z_Q = Z_{W_{0,q}}^{b_0} Z_{W_{1,q}}^{b_1}\cdots Z_{W_{r,q}}^{b_r} = [A^{(0)}(0)]^{b_0}[A^{(1)}(0)]^{b_1}[A^{(2)}(0)]^{b_2}\cdots[A^{(r)}(0)]^{b_r}. \qquad (17)$$

Examination of (13) and (17) (together with the multiplicativity of the power sums) shows that the power sums p_λ appearing in Z_Q will all have part sizes equal to a power of q and nothing else, and that the highest power of q appearing in a part size will be q^r. Actually, we can conclude a bit more.

Proposition 1. *Let q be prime and let Q be a Sylow-q-subgroup of $\mathfrak{S}_n$. Let $n = \sum_{i=0}^r b_i q^i$ be the q-adic expansion of n, where $b_r \neq 0$. If p_λ appears in Z_Q, then the part sizes of λ are powers of q. The smallest part size occurs as $[1]^n$, and the largest part size occurs as $[1]^{b_0}[q]^{b_1}\cdots[q^r]^{b_r}$. The coefficient of $p_{[1]^n}$ is $q^{-\operatorname{ord}_q(n!)}$, while that of $[1]^{b_0}[q]^{b_1}\cdots[q^r]^{b_r}$ is $((q-1)/q)^{\sum_{i=1}^r ib_i}$.*

3.3 Applications

To illustrate the main algorithm, we apply Theorem 1 in a few simple cases. The simplest case is $\mathfrak{S}_q$, in which case the Sylow-q-subgroups Q are all isomorphic to $\mathbb{Z}_q$. Applying Theorem 1 to (15) (and using (7)) we find

$$1_Q^{\mathfrak{S}_n} = \frac{1}{q}\sum_{\lambda}(\chi^\lambda([1]^q) + (q-1)\chi^\lambda([q]))\chi^\lambda \qquad (18)$$

and (using (1) and (5)),

$$
\begin{aligned}
1_Q^{\mathfrak{S}_n}(\nu) &= \frac{1}{q}(z_{[1]^q}\delta_{\lambda,[1]^q} + (q-1)z_{[q]}\delta_{\lambda,[q]}) \\
&= (q-1)!\delta_{\lambda,[1]^q} + (q-1)\delta_{\lambda,[q]}. \qquad (19)
\end{aligned}
$$

Next, let Q by a Sylow-q-subgroup of $\mathfrak{S}_{q^2}$. Applying Theorem 1 to (16) yields

$$
\begin{aligned}
1_Q^{\mathfrak{S}_{q^2}} = \sum_\lambda \Bigg(\frac{1}{q^{q+1}} \sum_{k=0}^{q} \binom{q}{k} (q-1)^{q-k}\chi^\lambda([1]^{qk}[q]^{q-k}) \\
+ \frac{(q-1)}{q^2}(\chi^\lambda([q]^q) + (q-1)\chi^\lambda([q^2])\Bigg) \chi^\lambda \qquad (20)
\end{aligned}
$$

and

$$
\begin{aligned}
1_Q^{\mathfrak{S}_{q^2}}(\nu) &= \frac{1}{q^{q+1}} \left(\sum_{k=0}^{q} \binom{q}{k} (q-1)^{q-k} z_{[1]^{qk}[q]^{q-k}} \delta_{\nu,[1]^{qk}[q]^{q-k}} \right) \qquad (21) \\
&\quad + \frac{q-1}{q^2}(z_{[q]^q}\delta_{\nu,[q]^q} + (q-1)z_{[q^2]}\delta_{\nu,[q^2]}) \\
&= \frac{q!}{q^{q+1}} \left(\sum_{k=0}^{q} \frac{(qk)!}{k!} (q(q-1))^{q-k} \delta_{\nu,[1]^{qk}[q]^{q-k}} \right) + q!(q-1)q^{q-2}\delta_{\nu,[q]^q} + (q-1)^2\delta_{\nu,[q^2]}. \qquad (22)
\end{aligned}
$$

As a last illustration of the method, we consider an example given in [7]. Let $q = 5$ and let Q be a Sylow-5-subgroup of $\mathfrak{S}_{38}$. Then

$$
\begin{aligned}
5^8 Z_Q = p_{[1]^{38}} &+ 28\,p_{[1]^{33}[5]} + 336\,p_{[1]^{28}[5]^2} + 2240\,p_{[1]^{23}[5]^3} + 8960\,p_{[1]^{18}[5]^4} \\
&+ 24004\,p_{[1]^{13}[5]^5} + 10000\,p_{[1]^{13}[25]} + 48672\,p_{[1]^8[5]^6} + 80000\,p_{[1]^8[5][25]} \\
&+ 56384\,p_{[1]^3[5]^7} + 160000\,p_{[1]^3[5]^2[25]}
\end{aligned}
$$

So, by Theorem 1 again,

$$
\begin{aligned}
5^8 1_Q^{\mathfrak{S}_{38}} = \sum_\lambda \{ &\chi^\lambda([1]^{38}) + 28\chi^\lambda([1]^{33}[5]) + 336\chi^\lambda([1]^{28}[5]^2) + 2240\chi^\lambda([1]^{23}[5]^3) \\
&+ 8960\chi^\lambda([1]^{18}[5]^4) + 24004\chi^\lambda([1]^{13}[5]^5) + 10000\chi^\lambda([1]^{13}[25]) \\
&+ 48672\chi^\lambda([1]^8[5]^6) + 80000\chi^\lambda([1]^8[5][25]) + 56384\chi^\lambda([1]^3[5]^7) \\
&+ 160000\chi^\lambda([1]^3[5]^2[25]) \} \chi^\lambda
\end{aligned}
$$

Similarly, one can obtain $1_Q^{\mathfrak{S}_{38}}(\nu)$.

References

[1] P. Diaconis, J. Fulman, and R. Guralnick, "On fixed points of permutations", *J. Alg. Comb.* **28** (2008) 189-218.

[2] E. Giannelli and S. Law, "On permutation characters and Sylow-p-subgroups of $\mathfrak{S}_n$", *J. Alg.* **506** (2018) 409-428.

[3] E. Giannelli and S. Law, "Sylow branching trees", arXiv:2409.07575.

[4] D. Goldschmidt, *Group Characters, Symmetric Functions, and the Hecke Algebra* (American Mathematical Society, Providence, 1993).

[5] L. Kaloujnine, "La stucture des p-groupes de Sylow des groupes symétriques finis", *Annales scientifiques de l'É.N.S., 3e série* **65** (1948) 239-276.

[6] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed. (Oxford University Press, Oxford, 1995).

[7] P. Renteln, "Counting Sylow double cosets in the symmetric group", *Disc. Math.* **347** (2024) 114109.

[8] G. de B. Robinson, *Representation Theory of the Symmetric Group* (Edinburgh University Press, Alva, Scotland, 1961).

[9] B. Sagan, *The Symmetric Group*, 2nd ed. (Springer, New York, 2001).

[10] R.P. Stanley, *Enumerative Combinatorics* Vol. 1, 2nd ed. (Cambridge, Cambridge University Press, 2011).

[11] R.P. Stanley, *Enumerative Combinatorics* Vol. 2 (Cambridge, Cambridge University Press, 1999).